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GOLDEN OLDIE EDITORIAL

Editorial note to: J. Ehlers, F. A. E. Pirani and A. Schild, The geometry of free fall and light propagation

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The article by Jürgen Ehlers, Felix Pirani, and Alfred Schild (EPS), reprinted here as a Golden Oldie, is devoted to the problem of deriving the Lorentzian geometry that underlies the space-time of general relativity from compatible conformal and projective structures on a four dimensional manifold. The geometry is based on a set of axioms; the proofs, even if not complete, are presented in a form appealing to physicists; they are well illustrated by carefully drawn figures. This article has been influenced—perhaps even inspired—by the early papers by Hermann Weyl on the foundations of differential geometry and their relation to physics. I use the opportunity of writing this note to briefly recall Weyl's articles [1,2] so as to emphasize the novelty of the approach and of the results presented in [3] reprinted here.

In Bernhard Riemann's approach to geometry (1854), everything was founded on the metric tensor g. The Christoffel symbols[1]

$$\Gamma(g)^{\mu}_{\nu\rho} \overset{\text{def}}{=} \tfrac{1}{2} g^{\mu\sigma}(g_{\sigma\nu,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma})$$

[1] In this note, Greek letters are used to denote space-time indices: $\mu, \nu, \rho = 1, 2, 3, 4$; EPS use Latin indices. A comma followed by an index denotes differentiation with respect to the corresponding coordinate.

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were auxiliary quantities convenient to write the geodetic equation, obtained by minimizing the length integral. Tullio Levi-Civita's [4] parallel transport of vectors was determined by the metric induced on a manifold by its embedding in a flat space.

Gerhard Hessenberg [5] was the first to introduce what is now called a linear connection compatible with the metric, but not necessarily symmetric. His asymmetric connection implicitly contained torsion. He also distinguished the *shortest lines* (extremals of $\int \mathrm{d}s$) from the *straight lines* (autoparallels) and pointed out that these notions coincide for connections that are symmetric and metric. Élie Cartan properly defined torsion of a linear connection and suggested its possible physical role [6,7].

The first section of Weyl's article [1] is entitled *On the relation between geometry and physics*. Weyl distinguished three levels of building the geometry: the *continuum*—a bare manifold with smooth structure only—corresponds to an empty world, an *affine* (linear) *connection* describes gravitation and a *metric* structure underlies the 'æther', he said. Weyl imposed on the connection a condition of 'commutativity', equivalent to its symmetry. The metric structure, as defined in this paper by Weyl, was a *conformal geometry*, given by an equivalence class $\mathscr{C}$ of metrics g on a manifold M and a connection compatible with $\mathscr{C}$. In contemporary notation, such a *Weyl geometry* [8] is a pair $(\mathscr{C}, \nabla)$, where ∇ denotes the covariant derivative corresponding to a connection Γ, i.e. $\nabla_\nu X^\mu \overset{\text{def}}{=} X^\mu{}_{,\nu} + \Gamma^\mu_{\nu\rho} X^\rho$ for a vector field X. It is worth noting that Weyl uses the expression *null* lines rather than *isotropic*, a misnomer introduced by pure mathematicians; there are remarks on its origin in [9].

The equivalence relation, defining the class $\mathscr{C}$, is

$$g \overset{\mathscr{C}}{\equiv} g' \Leftrightarrow \text{there is a function } f \text{ on } M \text{ such that } g' = (\exp f)g. \tag{A}$$

Compatibility between $\mathscr{C}$ and ∇ means that, for every $g \in \mathscr{C}$, there is a one-form $A = A_\mu \mathrm{d}x^\mu$ such that

$$\nabla_\mu g_{\nu\rho} = A_\mu \, g_{\nu\rho}. \tag{B}$$

A consequence of (A) and (B) is $\nabla_\mu g'_{\nu\rho} = A'_\mu \, g'_{\nu\rho}$, where $A'_\mu = A_\mu + f_{,\mu}$. Weyl identified the 2-form $F = \mathrm{d}A$ with the electromagnetic field. With the advent of quantum theory, the transformations involving f were suitably reinterpreted and gave rise to the *dawning of gauge theory* [10]. Recent work on Weyl geometry and its applications in relativistic physics has been reviewed in [11].

In [2] Weyl considers, in addition to $\mathscr{C}$, a *projective structure* defined as an equivalence class $\mathscr{P}$ of symmetric connections Γ. The equivalence relation, defining the class $\mathscr{P}$, is

$$\Gamma \overset{\mathscr{P}}{\equiv} \Gamma' \Leftrightarrow \exists \text{ one-form } \psi_\nu \mathrm{d}x^\nu \text{ such that } \Gamma'^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho} + \delta^\mu_\nu \psi_\rho + \delta^\mu_\rho \psi_\nu. \tag{C}$$

The class $\mathscr{P}$ defines a family of geodesics without a preferred parametrization. The direction of a vector Y that undergoes parallel transport along the trajectory of a vector field X changes when Γ is replaced by Γ', unless $Y \| X$. This direction does not change when Γ is replaced by $\Gamma''^{\mu}_{\nu\rho} = \Gamma^{\mu}_{\nu\rho} + \delta^{\mu}_{\rho}\psi_{\nu}$. The replacement $\Gamma \mapsto \Gamma''$ does not, however, respect the symmetry of connections. The class $\mathscr{S}$ of connections defined by the equivalence relation $\Gamma \overset{\mathscr{S}}{\equiv} \Gamma''$ (for some ψ) determines the same family of geodesics as $\mathscr{P}$, provided that $\mathscr{P} \cap \mathscr{S} \neq \varnothing$. Such properly projective transformations of connections, considered already by J. M. Thomas [12], play a role in the Einstein–Cartan theory [13].

If g' is as in (A), then its Christoffel symbols are

$$\Gamma(g')^{\mu}_{\nu\rho} = \Gamma(g)^{\mu}_{\nu\rho} + \tfrac{1}{2}(\delta^{\mu}_{\nu} f_{,\rho} + \delta^{\mu}_{\rho} f_{,\nu} - g^{\mu\sigma} g_{\nu\rho} f_{,\sigma}). \tag{D}$$

The first theorem (*Satz 1*) in [2] reads: *The projective and conformal structures of a metric space determine uniquely its metric*. The simple proof of *uniqueness* of g, up to a constant scale factor, given by Weyl, is as follows: if g and $g' = (\exp f)g$ are conformally related metrics, corresponding to the same projective structure, $\Gamma(g)$ and $\Gamma(g') \in \mathscr{P}$, then there hold equations (C) (with $\Gamma = \Gamma(g)$ and $\Gamma' = \Gamma(g')$) and (D). By equating the right sides of those two equations, and using a simple algebraic argument, one obtains $\mathrm{d}f = 0$ so that the metrics are related by a constant scalar factor. Weyl, however, considers neither the problem of *existence* of such a g, nor the question of compatibility between $\mathscr{C}$ and $\mathscr{P}$. He emphasizes though that the metric tensor can be obtained by observation of the motion of material particles and of the propagation of light. In the rest of the paper, Weyl introduces a new concept: the *projective curvature tensor*, describes its properties, as well as those of the tensor of conformal curvature, with references to Cotton and Schouten [14].

Let me now turn to the Golden Oldie by Ehlers, Pirani and Schild. In the first section, the authors convincingly argue in favor of a deductive approach to the foundations of space-time geometry, based on the use of light rays and freely falling particles, rather than on clocks of J. L. Synge's [15,16] chronometric approach. The propagation of light defines a conformal structure $\mathscr{C}$ of Lorentzian signature and freely falling particles determine the projective structure $\mathscr{P}$. The conformal structure defines the notions of time-like, null, and space-like vectors. Therefore, one can also introduce the notion of null projective geodesics: they are geodesics of $\mathscr{P}$ with null tangent vectors. The structures $\mathscr{C}$ and $\mathscr{P}$ are required to be *compatible*. This is an important, necessary condition for the existence of a Lorentzian metric underlying both structures, a condition that Weyl seems to have missed. It is expressed by the requirement that

$$\text{all null geodesics of } \mathscr{C} \text{ are also geodesics of } \mathscr{P}. \tag{E}$$

The authors call a manifold endowed with such a compatible pair $(\mathscr{C}, \mathscr{P})$ a *Weyl space*, a notion close to, but not identical with, that of Weyl geometry described above. They state that a Weyl space has a unique *affine* structure $\mathscr{A}$ such that its geodesics coincide with the geodesics of $\mathscr{P}$ and the property of vectors to be null with respect to $\mathscr{C}$ is preserved by the parallel transport defined by $\mathscr{A}$.

In paragraph (e) on p. 69 they formulate additional assumptions on the geometry of a Weyl space $(\mathscr{C}, \mathscr{P})$ so that it can be derived from a Riemannian metric $g \in \mathscr{C}$ such that $\Gamma(g) \in \mathscr{P}$. They are formulated as conditions on the curvature tensor associated with the affine structure $\mathscr{A}$.

In Sect. 2, EPS formulate four groups of axioms, underlying space-time geometry. The first concerns its structure as a smooth manifold, and the second—the light propagation and conformal structure. To describe the latter, they introduce a tensor density[2] $\mathfrak{g}$ of signature $(+++-)$,

$$\mathfrak{g}_{\mu\nu} = g_{\mu\nu}/|\det(g_{\rho\sigma})|^{1/4} \quad \text{so that} \quad \det(\mathfrak{g}_{\mu\nu}) = -1.$$

The density $\mathfrak{g}$ has weight $-1/2$ and is invariant with respect to $g \mapsto (\exp f)g$ so that the single field $\mathfrak{g}$ completely characterizes $\mathscr{C}$. The density $\mathfrak{g}_{\mu\nu}$ and its inverse $\mathfrak{g}^{\mu\nu}$ are used to lower and raise indices. The third group of axioms deals with the projective structure $\mathscr{P}$; they choose $\Pi \in \mathscr{P}$ such that $\Pi^{\nu}_{\mu\nu} = 0$. The very important axiom **C** (p. 78) concerns compatibility between $\mathscr{P}$ and $\mathscr{C}$. The coefficients of a conformal connection[3] are defined as $K = \Gamma(\mathfrak{g})$. On the basis of axiom **C**, the authors derive the equation

$$\Pi^{\mu}_{\nu\rho} - K^{\mu}_{\nu\rho} = 5q^{\mu}\mathfrak{g}_{\nu\rho} - 2\delta^{\mu}{}_{(\nu}q_{\rho)} \quad \text{for some vector } q^{\mu} = \mathfrak{g}^{\mu\nu}q_{\nu}.$$

It implies that if u is a *null* vector, then

$$(\Pi^{\mu}_{\nu\rho} - K^{\mu}_{\nu\rho})u^{\nu}u^{\rho} \| u^{\mu},$$

so that null geodesics of $\mathscr{C}$ and $\mathscr{P}$ coincide and the compatibility condition is satisfied.

The additional assumptions described in part (e) of the Introduction are used, in the last paragraph of the article, entitled *Curvature and Riemannian space-time*, to argue that the structure of a Weyl space leads to a Lorentzian metric, defined up to a constant scalar factor. Essentially, a *Riemannian axiom* (p. 82) reduces to the statement that the curvature tensor of the affine connection $\mathscr{A}$ is a 2-form with values in the Lie algebra of the Lorentz group. One can then appeal to the theorems on holonomy to construct a reduction of the bundle of frames to (a subgroup of) the Lorentz group; this is known to be equivalent to the existence of a Lorentzian metric [20,21].

The authors frankly say, on pages 69 and 70, that *'A fully rigorous formalization* [of our axioms and proofs] *has not yet been achieved, but we nevertheless hope that the main line of reasoning will be intelligible and convincing to the sympathetic reader.'* In particular, the paper does not contain a formal proof of the existence, in a Weyl space, of the affine structure $\mathscr{A}$ with the properties announced in the Introduction (p. 67).

In connection with the EPS article it is worth to recall the old paper [22] where Élie Cartan emphasizes the connection between conformal geometry and physics (the *optical universe*), defines *normal conformal spaces* with a vanishing Ricci tensor and

[2] To denote this density, EPS use a script g letter in a font that is not available in LaTeX.

[3] A word of warning: mathematicians give to the expressions 'conformal connection' and 'projective connection' a different meaning than the one used by EPS; see [17,18] and, for a contemporary treatment, ch. IV in [19].

states that the tensor of conformal curvature of a Lorentzian space defines four null directions, thus providing the geometric basis for the later Petrov classification in Penrose's form; see also the remark in footnote on p. 164 of [23].

The results of [3] have been extended in [24]: Ehlers and Schild gave there a projective analogue of the geodetic deviation equation that can be used for a geometric characterization of projective curvature.

In view of its fundamental character and importance, the EPS paper has attracted, over the years, considerable interest and led to many publications. Here is only a sample of references: [25–28]. There is also a recent survey by John Stachel [29]. I think it is clear that the subject requires and deserves further work.

I am grateful to Engelbert Schücking and Friedrich Hehl for having drawn my attention to the paper of Hessenberg and explained its significance. Friedrich gave me also very precious advice on the literature related to the subject of the EPS paper. I thank Ilka Agricola and Erhard Scholz for illuminating remarks.

Comment by the Golden Oldies Editor: Short biographies of the authors of the paper were published in our journal on other occasions, namely:

1. Jürgen Ehlers: Gen. Relativ. Gravit. **41**(9), 1899 (2009), doi:10.1007/s10714-009-0841-7.
2. Felix Pirani: Gen. Relativ. Gravit. **41**(5), 1199 (2009), doi:10.1007/s10714-009-0785-y.
3. Alfred Schild: Gen. Relativ. Gravit. **8**(11), 955 (1977), doi:10.1007/BF00759241.

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